PHYS371: Classical Mechanics

Equations & Constants

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Constants

- Speed of light (in vacuum): $c = 2.998 \times 10^8\,\mathrm{m\,s^{-1}}$
- Gravitational const.: $G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
- Table:

Object (symb.)	Mass (kg)	Radius (m)
Earth ($\oplus$)	5.972×10^{24}	6.378×10^{6}
Moon ($☾$)	7.394×10^{22}	1.738×10^{6}
Sun ($\odot$)	1.989×10^{30}	6.957×10^{8}
Electron (e^-)	9.109×10^{-31}	$< 10^{-16}$
Proton (p^+)	1.673×10^{-27}	$\approx 10^{-15}$
Neutron (n^0)	1.675×10^{-27}	$\approx 10^{-15}$

- Earth-Sun distance (astronomical unit): $1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$
- Earth-Moon distance: $3.844 \times 10^6\,\mathrm{m}$
- Pressure: $1\,\mathrm{Pa}$ (pascal) $= 1\,\mathrm{N\,m^{-2}}$
  $1\,\mathrm{Pa} = 9.869 \times 10^{-6}\,\mathrm{atm}$ (atmosphere)
- Planck constant: $h = 6.626 \times 10^{-34}\,\mathrm{J\,s}$
- Boltzmann constant: $k_\mathrm{B} = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
- Permittivity of vacuum: $\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$
- Fundamental quantum of charge: $e = 1.602 \times 10^{-19}\,\mathrm{C}$
- Coulomb constant: $k_e = \dfrac{1}{4\pi\varepsilon_0} = 8.988 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$

Equations

- Taylor series: $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(a)$ denotes nth derivative of f evaluated at point a. Factorial of zero is one (i.e., $0! = 1$).
- Cylindrical-Polar Coordinates
$$\begin{aligned}\vec{\mathbf{v}} &= \dot{s}\,\hat{\mathbf{s}} + s\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{k}}\\ \vec{\mathbf{a}} &= \left(\ddot{s} - s\dot{\phi}^2\right)\hat{\mathbf{s}} + \left(s\ddot{\phi} + 2\dot{s}\dot{\phi}\right)\hat{\boldsymbol{\phi}} + \ddot{z}\,\hat{\mathbf{k}}\end{aligned}$$
- Spherical Coordinates
$$\begin{aligned}\vec{\mathbf{v}} &= \dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}\\ \vec{\mathbf{a}} &= \left(\ddot{r} - r\dot{\phi}\sin^2\theta - r\dot{\theta}^2\right)\hat{\mathbf{r}} +\\ &\quad \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\hat{\boldsymbol{\theta}} +\\ &\quad \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta\right)\hat{\boldsymbol{\phi}}\end{aligned}$$
- Differential equation of an orbit in central field: $\dfrac{d^2u}{d\phi^2} + u = -\dfrac{1}{ml^2u^2}f(u^{-1})$, where l is the angular momentum per unit mass.
- Electric or Coulomb force: $\vec{\mathbf{F}}_{12}(r) = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1Q_2}{r^2}\,\hat{\mathbf{r}}_{12}$, where $Q_\#$ are charges.
- Hooke's law: $\vec{\mathbf{F}} = -k(\vec{\mathbf{r}} - \vec{\mathbf{r}}_\mathrm{eq})$, where k is spring constant and $\vec{\mathbf{r}}_\mathrm{eq}$ is its equilibrium position.
- Damping factor: $\gamma = \dfrac{c}{2m}$ and
  q-factor: $q = \sqrt{\gamma^2 - \omega_0^2}$
- Drag force for large object and moderate speed in fluid: $\vec{\mathbf{F}}_\mathrm{D} = (-C\rho A v^2/2)\,\hat{\mathbf{v}}$, where C is drag coefficient, A is area, and ρ is density of fluid.
- Stoke's law (drag force for small object in denser fluid): $\vec{\mathbf{F}}_\mathrm{S} = -6\pi r\eta\vec{\mathbf{v}}$, where r is radius and η is viscosity of fluid.
- Work: $W_\mathrm{AB} = \displaystyle\int_{\substack{\text{path}\\ \mathrm{A}\to\mathrm{B}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$
- Power: $P = \dfrac{dW}{dt}$
- Impulse: $\vec{\mathbf{J}} = \displaystyle\int_{t_0}^{t_\mathrm{f}} \vec{\mathbf{F}}(t)dt = \vec{\mathbf{p}}_\mathrm{f} - \vec{\mathbf{p}}_0$
- Angular velocity: $\vec{\boldsymbol{\omega}}$, where $\vec{\mathbf{v}}_\mathrm{tan} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$
- Angular acceleration: $\vec{\boldsymbol{\alpha}}$, where $\vec{\mathbf{a}}_\mathrm{tan} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}}$
- Torque: $d\vec{\boldsymbol{\tau}} = d(\vec{\mathbf{r}} \times \vec{\mathbf{F}})$
- Centripetal acceleration: $\vec{\mathbf{a}}_\mathrm{cent} = -\dfrac{v_\mathrm{tan}^2}{r}\,\hat{\mathbf{r}} = -\omega^2\vec{\mathbf{r}}$
- Transforming from moving and rotating system to fixed system:
$$\begin{aligned}\vec{\mathbf{v}} &= \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}' + \vec{\mathbf{v}}_0\\ \vec{\mathbf{a}} &= \vec{\mathbf{a}}' + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\mathbf{r}}' + 2\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}') + \vec{\mathbf{a}}_0\end{aligned}$$
  from operator: $\left[\left(\dfrac{d}{dt}\right)_\mathrm{rot} + \vec{\boldsymbol{\omega}}\,\times\right]$
- Projectile motion in a rotating coordinate system
$$\begin{aligned}x'(t) &= \frac{1}{3}\omega g t^3\cos\lambda - \omega t^2\left(\dot{z}_0'\cos\lambda - \dot{y}_0'\sin\lambda\right) + \dot{x}_0't + x_0'\\ y'(t) &= \dot{y}_0't - \omega\dot{x}_0't^2\sin\lambda + y_0'\\ z'(t) &= -\frac{1}{2}gt^2 + \dot{z}_0't + \omega\dot{x}_0't^2\cos\lambda + z_0'\end{aligned}$$
- Lagrange's equations:
$$\frac{\partial L}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = 0 \text{ for } n = 1, 2, ..., \mathcal{N}$$
- Moment of inertia for continuous objects:
  - Hoop about cylinder axis (C.A.): $I = MR^2$
  - Hoop about any diameter: $I = \frac{1}{2}MR^2$; same as solid cylinder/disk about C.A.
  - Annular cylinder/ring about C.A: $I = \frac{1}{2}M(R_1^2 + R_2^2)$
  - Solid cylinder/disk about central diameter: $I = \frac{1}{4}MR^2 + \frac{1}{12}M\ell^2$
  - Thin rod about central diameter ($\perp$): $I = \frac{1}{12}M\ell^2$; same as thin rectangle with side ℓ about center parallel to other side
  - Thin rod about end ($\perp$): $I = \frac{1}{3}M\ell^2$
  - Solid sphere about any diameter: $I = \frac{2}{5}MR^2$
  - Thin spherical shell about any diameter: $I = \frac{2}{3}MR^2$
  - Slab about $\perp$ axis through center: $I = \frac{1}{12}M(a^2 + b^2)$
- Parallel-Axis Theorem: $I_{\parallel-\mathrm{axis}} = I_\mathrm{CM} + Md^2$

- Moment of inertia about arbitrary axis: $I = \tilde{\boldsymbol{n}}\boldsymbol{I}\boldsymbol{n}$ (matrix mult.), where matrix:

$$\boldsymbol{I} = \sum_{u\,\text{comp.}} \left(I_{xu}\,\hat{\mathbf{i}} + I_{yu}\,\hat{\mathbf{j}} + I_{zu}\,\hat{\mathbf{k}} \right);$$

moments of inertia:

$$\begin{aligned} I_{xx} &= \int (y^2 + z^2)dm \\ I_{yy} &= \int (z^2 + x^2)dm \\ I_{zz} &= \int (x^2 + y^2)dm; \end{aligned}$$

products of inertia:

$$\begin{aligned} I_{xy} = I_{yx} &= -\int xydm \\ I_{yz} = I_{zy} &= -\int yzdm \\ I_{zx} = I_{xz} &= -\int zxdm; \end{aligned}$$

and angles α, β, and γ are between $\vec{\boldsymbol{\omega}}$ and x, y, and z, respectively.

- Angular momentum: $d\vec{\mathbf{L}} = d(\vec{\mathbf{r}} \times M\vec{\mathbf{v}})$; $\dfrac{d\vec{\mathbf{L}}}{dt} = \sum_{n=1}^{\mathcal{N}} \vec{\boldsymbol{\tau}}_n$; and $\vec{\mathbf{L}} = \boldsymbol{I}\vec{\boldsymbol{\omega}}$
- Rotational kinetic energy: $K_{\text{rot}} = \dfrac{1}{2}\vec{\boldsymbol{\omega}} \bullet \vec{\mathbf{L}}$
- Doppler shift: $\dfrac{v}{c} = \dfrac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$
- General 1D wave equation: $x(t) = A\cos(\omega t + \phi)$
- Harmonic condition: $\nabla^2\psi = \dfrac{1}{v^2}\dfrac{\partial^2\psi}{\partial t^2}$
- Photon energy: $E = hf$
- Newton's generalization of Kepler's 3rd Law: $T^2 = \dfrac{4\pi^2}{G(M_1 + M_2)}\,a^3$, where T is orbital period and a is total distance between centers of M_1 and M_2.
- Lagrange's equation:

$$\frac{\partial L}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) + \sum_j \lambda_j \frac{\partial f_j}{\partial q_n}$$

- Hamiltonian

$$H = \sum_n \dot{q}_n p_{q_n} - L$$

- Hamilton's equations:

$$\frac{\partial H}{\partial p_{q_n}} = \dot{q}_n \text{ and } \frac{\partial H}{\partial q_n} = -\dot{p}_{q_n}$$

Vectors

- Scalar/dot product: $\vec{\mathbf{A}} \bullet \vec{\mathbf{B}} = AB\cos\theta$, where θ is angle between, or: $\vec{\mathbf{A}} \bullet \vec{\mathbf{B}} = \sum_{u\ \text{comp.}} A_u B_u$
- Vector/cross product: $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB\sin\theta$, in direction $\perp$ to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, or: $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \sum_{u\ \text{comp.}} \sum_{w\ \text{comp.}} A_u B_w (\hat{\mathbf{u}} \times \hat{\mathbf{w}})$

Geometry

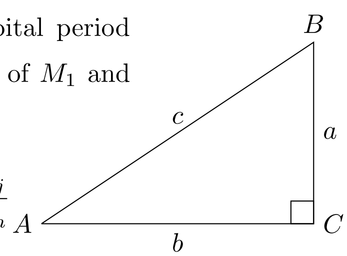
- Surface area of a sphere: $A = 4\pi r^2$
- Volume of a sphere: $V = \dfrac{4}{3}\pi r^3$
- Surface area of a cone: $A = \pi r\left(r + \sqrt{h^2 + r^2}\right)$
- Volume of a cone: $V = \dfrac{1}{3}\pi r^2 h$
- Conic sections (polar coordinates)
  - All cases: $r = \dfrac{r_0(1+\epsilon)}{1 + \epsilon\cos\theta}$, where focus is at origin; r_0 is minimum value of $|r|$
  - Ellipse: $\epsilon < 1$, Parabola $\epsilon = 1$, Hyperbola $\epsilon > 1$, Circle $\epsilon = 0$.
  - For ellipse, semimajor axis: $a = \dfrac{r_0}{1-\epsilon}$ and semiminor axis: $b = a\sqrt{1-\epsilon^2}$.

Calculus

- $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(e^{ax}) = ae^{ax}$
- Laplacian operator: $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$
- "Del" (or "nabla") operator: $\vec{\boldsymbol{\nabla}} = \dfrac{\partial}{\partial x}\,\hat{\mathbf{i}} + \dfrac{\partial}{\partial y}\,\hat{\mathbf{j}} + \dfrac{\partial}{\partial z}\,\hat{\mathbf{k}}$

Trigonometry (see figure below)

- Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos(\angle C)$
- Law of Sines: $\dfrac{a}{\sin(\angle A)} = \dfrac{b}{\sin(\angle B)} = \dfrac{c}{\sin(\angle C)}$
- $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$
- $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$
- $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
- $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$



Non-SI units

- Inch: 1 in = 2.540 cm
- Foot: 1 ft = 12 in = 0.305 m
- Mile: 1 mi = 5280 ft = 1.609 km
- Pound: 1 lb = 16 oz = 0.454 kg
- Gallon: 1 gal = 128 fl oz = 3.785 L